Finite Groups

(ref: Fulton-Harris, Representation Theory, Part I, Ch1-4, 6).

Chapter 1. Repr. of finite groups.

$|G| < \infty$ (same for compact Lie group)

$\rightsquigarrow$ sum / integration

$\Rightarrow \forall\ G \longrightarrow GL(V)$
$\qquad\qquad\qquad\quad \cup\!|$
$\qquad\qquad\qquad\quad U(V) \quad (\exists\ h)$

$\Rightarrow$ complete reducibility / semisimplicity.

Schur's lemma. $G \curvearrowright V, W$ irred $\quad \varphi : V \xrightarrow{\text{G-map}} W$

$\Rightarrow$ (i) $\varphi \ \cong$ or 0

(ii) $V = W \Rightarrow \varphi = \lambda I \quad \exists\ \lambda \in \mathbb{C}$.

Eg. G Abelian. $\quad G \curvearrowright V$ irred $\Rightarrow$ 1 dim.

$G^\vee := \{ \rho : G \xrightarrow{\text{homo.}} \mathbb{C}^\times \} \simeq \{ \text{irred. rep. of } G \}$

Eg. S_n permutation group / symmetric group.

- $S_n \curvearrowright \mathbb{C} = U$ trivial $\quad$ • $S_n \curvearrowright U'$ 1 dim. $\quad g \cdot v = \operatorname{sgn}(g) v$.

- $S_n \curvearrowright \mathbb{C}^n \ \underline{\underline{\text{G-irred. decomp.}}}\ \mathbb{C}\langle(1,1,\cdots 1)\rangle \oplus \underbrace{\{(z_1,\dots,z_n) \mid \sum_{i=1}^n z_i = 0\}}_{V}$ std. repr

(permute coordi.)

$S_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^3 = 1, \sigma\tau\sigma = \tau^2 \rangle \curvearrowright V = \mathbb{C}\langle \alpha, \beta \rangle$

$\sigma = (12)$, $\tau = (123)$, $\alpha = (\omega\ 1\ \omega^2)$, $\beta = (1\ \omega\ \omega^2)$

w/ $\tau\alpha = \omega\alpha$, $\tau\beta = \omega^2\beta$, $\sigma\alpha = \beta$, $\sigma\beta = \alpha$

U, U', W : ALL irred. repr. of S_3.

Chapter 2. Characters.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\chi_V"'] & GL(V) \arrow[d, "Tr"] \\ & \mathbb{C} \end{tikzcd}$$

- $\chi_V(hgh^{-1}) = \chi_V(g)$
  i.e. class function

- $\chi_{V\oplus W} = \chi_V + \chi_W$
  $\chi_{V\otimes W} = \chi_V \cdot \chi_W$
  i.e. $\{G\text{-mod.}\} \longrightarrow C^\infty(G)^{Ad(G)} = \mathbb{C}[G]^{AdG}$
  alg. homo.

- $\chi_{V^*} = \overline{\chi_V}$
  $\chi_{\Lambda^2 V}(g) = (\chi_V(g)^2 - \chi_V(g^2))/2$ $\quad \left(\because \sum_{i<j} \lambda_i\lambda_j = \frac{(\sum\lambda_i)^2 - \sum\lambda_i^2}{2}\right)$
  $\chi_{Sym^2 V}(g) = (\quad " \quad + \quad " \quad)/2$

(Note: $V\otimes V = \Lambda^2 V \oplus Sym^2 V$).

Eg. character table for

S_3 :	1	(12)	(123)
	1	3	2 ← # of elt. in conj class.
U	1	1	1
U'	1	−1	1
V	2	0	−1

- U, U', V : all irred S_3-mod
- $\chi_U, \chi_{U'}, \chi_V$: indep.

$\Rightarrow$ ANY G-mod W is determinded by χ_W.

§. $\forall\ G \longrightarrow GL(V)$

Let $\varphi \triangleq \frac{1}{|G|}\sum_{g\in G} g \in End(V) =: \mathfrak{gl}(V)$

- $\varphi : V \longrightarrow V$ is projection to $V^G := \{v\in V : G\cdot v = v\}$

- $\underbrace{Tr\,\varphi}_{\frac{1}{|G|}\sum_{g\in G}\chi_V(g) =: \int_G \chi_V} = \dim V^G$

Theorem: $\mathbb{C}[G]^{\mathrm{Ad}\,G}$, $\langle \alpha, \beta \rangle := \int_G \bar{\alpha}\beta$ inner product

$\cup\!\!\mid$

$\{\chi_V\text{'s}\}_{V \text{ irred}}$: orthonormal [base]

Pf. $\langle \chi_V, \chi_W \rangle = \int_G \bar{\chi}_V \cdot \chi_W = \int_G \chi_{\mathrm{Hom}(V,W)}$

$= \dim(\underbrace{\mathrm{Hom}(V,W)^G}_{\{G\text{-homo.}\}}) \underset{\text{Schur.}}{=} \begin{cases} 1 & V \simeq W \\ 0 & V \not\simeq W. \end{cases}$

Cor. $\#\{\text{irred } G\text{-mod}\} \leq \underbrace{\#\{\text{conj. classes}\}}_{\dim \mathbb{C}[G]^{\mathrm{Ad}\,G}}$ [=]

Cor. V determined by χ_V

V irred $\Leftrightarrow \langle \chi_V, \chi_V \rangle = 1$

V_i irred, $\Rightarrow$ multi. of V_i in $V = \langle \chi_{V_i}, \chi_V \rangle$

Eg. $G \curvearrowright \mathbb{C}[G] =: R$ regular repr.

$\chi_R(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$

$\forall$ irred. $V_i \Rightarrow \langle \chi_{V_i}, \chi_R \rangle = \frac{1}{|G|} \chi_{V_i}(e) \cdot |G| = \dim V_i$

$\Rightarrow$ V_i appears in R $\dim V_i$ - times.

[$\mathbb{C}[G] = \bigoplus_{V_i:\text{irred.}} \mathrm{End}\, V_i$ as $G_L \times G_R$ - mod.].

Cor. $|G| = \dim R = \sum_{V_i} (\dim V_i)^2$

$0 = \sum_{V_i} (\dim V_i) \cdot \chi_{V_i}(g) \quad \forall\, g \neq e$

Eg.

S_4	1	$(1\,2)$	$(1\,2\,3)$	$(1\,2\,3\,4)$	$(12)(34)$
	1	6	8	6	3
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
$V \otimes U'$	3	-1	0	1	-2
missing W	2	0	-1	0	2

$2 = \dim W \Rightarrow \sigma = 1$

$24 = |G| = 1^2 + 1^2 + 3^2 + 3^2 + 2^2$ (Cor)

$\sigma := (12)(34)$ $\quad \sigma^2 = 1$

$\Rightarrow$ decend to $S_4 / \mathrm{Ad}G \cdot \sigma \simeq S_3$

$\Rightarrow$ $W =$ std. rep. of S_3 !

§ Prop $\alpha \in \mathbb{C}[G]$

$\alpha \in \mathbb{C}[G]^{\mathrm{Ad}G} \iff \forall\, G\text{-mod. } V,\ \varphi_{\alpha,V}(g) \triangleq \sum_{g \in G} \alpha(g) \cdot g : V \longrightarrow V$ G-homo.

In this situation, if V irred $\overset{\text{Schur}}{\Longrightarrow} \varphi_{\alpha,V} = \lambda I$

$$\lambda \cdot \dim V = \mathrm{Tr}\, \varphi_{\alpha,V} = \sum_{g \in G} \alpha(g) \cdot \underbrace{\mathrm{Tr}_V\, g}_{\chi_V(g)} = \overline{(\alpha, \chi_{V^*})}$$

Suppose, $\alpha \perp \chi_V$ $\quad \forall$ irred. V

$\Rightarrow \lambda = 0$

$\Rightarrow 0 = \varphi_{\alpha,V}(g) = \sum_{g \in V} \alpha(g) \cdot g : V \to V$ $\quad \forall$ irred. V

$\Rightarrow 0 = (\text{—} \ '' \ \text{—})$ $\forall$ repr., say R

$\Rightarrow \alpha \equiv 0$ $\quad$ ($\because$ g's l.i. in R)

$\Rightarrow$ Theorem:

- $\#$ irred. rep. of G $=$ $\#$ conj. classes of G
- $\{ \chi_V\text{'s} \}_{V:\text{irred}} \subseteq \mathbb{C}[G]^{\mathrm{Ad}G}$: o.n. basis.

Repr. ring:

$V - W \in R(G) \xrightarrow{\chi} \mathbb{C}[G]^{\mathrm{Ad}G}$ $\quad$ injective isometry.

§ $S_d \curvearrowright V \simeq \mathbb{C}^{d-1}$ std. rep.

- $S_d \curvearrowright \Lambda^k V$ irred. $\forall k$

[Pf: $\Lambda^k \mathbb{C}^d = \Lambda^k(U \oplus V) = \Lambda^k V \oplus \Lambda^{k-1} V$

enough to show $(\chi_{\Lambda^k V}, \chi_{\Lambda^k V}) = 2$ ✓ by combinatoric

- $S_d \curvearrowright \mathrm{Sym}^k V$ almost never irred.

§ Induced repr.

$$G \geq H \curvearrowright W \rightsquigarrow G \curvearrowright V := \mathrm{Ind}_H^G W$$

$$= \bigoplus_{\sigma \in G/H} \sigma . W$$

$$= \Gamma\left(\begin{array}{c} G \times_H W \\ \downarrow \\ G/H \end{array} \circlearrowright \right)$$

$$\begin{tikzcd}
H \arrow[d] \arrow[r, phantom, "\curvearrowright"] & W \arrow[d] \\
G \arrow[d] & G \times_H W \arrow[d] \\
G/H \arrow[r, phantom, "="] & G/H \\
\text{pr. H-bdl.} & \text{assoc. VB}
\end{tikzcd}$$

- Frobenius reciprocity.

$H \leq G$, $H \curvearrowright W$, $G \curvearrowright U$ $\Rightarrow$ $(\chi_W, \chi_{\mathrm{Res}\, U})_H = (\chi_{\mathrm{Ind}\, W}, \chi_U)_G$

say W, U irred $\Rightarrow$ # times W in $\mathrm{Res}\, U$ = # times U in $\mathrm{Ind}\, W$

To prove it, enough to show

any $W \xrightarrow{\text{H-map}} U$ extend (uniquely) to $\mathrm{Ind}\, W \xrightarrow{\text{G-map}} U$

i.e. $\mathrm{Hom}_H(W, \mathrm{Res}\, U) \simeq \mathrm{Hom}_G(\mathrm{Ind}\, W, U)$.

Eg. $V_2 = U_2' \curvearrowleft S_2 \leq S_3 \xRightarrow{\text{Reciprocity}} \mathrm{Ind}\, V_2 \simeq U_3' \oplus V_3$

Eg. $V_3 \curvearrowleft S_3 \leq S_4 \xRightarrow{\text{reciprocity}} \mathrm{Ind}\, V_3 \simeq V_4 \oplus V_4' \oplus W$.

§ Group algebra $\mathbb{C}[G] = L^2(G)$ w/ $e_g . e_h = e_{gh}$

$$\mathbb{C}[G] \overset{\text{alg.}}{\simeq} \bigoplus_{V_i \text{ irred.}} \mathrm{End}\, V_i$$

Chapter 4. Repr. of S_d

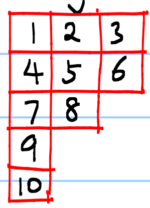
- $|S_d| = d!$
- conj. class $\longleftrightarrow$ partition of $d = \lambda_1 + \dots + \lambda_k$, $\lambda_1 \geq \dots \geq \lambda_k \geq 1$

  $p(d) :=$ # of ——— " ———
- $\sum_{d=0}^{\infty} p(d) t^d = \prod_{n=1}^{\infty} \frac{1}{1-t^n}$

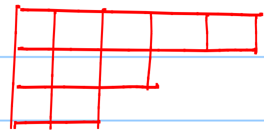
(Recall #conj. class = # irred. repr.)

Eg. S_{10} $\quad 10 = 3+3+2+1+1$

$\rightsquigarrow$ Young diagram



$\rightsquigarrow$ Conjugate



$10 = 5+3+2$

$P_\lambda := \{ g \in S_d : g \text{ preserves rows} \}$ $\quad a_\lambda \triangleq \sum_{g \in P_\lambda} g \in \mathbb{C}[S_d]$

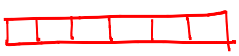
$Q_\lambda := \{$ ——— " ——— columns $\}$ $\quad b_\lambda \triangleq \sum_{g \in Q_\lambda} g \in \mathbb{C}[S_d]$

$c_\lambda \triangleq a_\lambda \cdot b_\lambda \in \mathbb{C}[S_d]$

Theorem: (1) $c_\lambda^2 = n_\lambda c_\lambda$

(2) $V_\lambda := \mathbb{C}[S_d] \cdot c_\lambda$ irred. G-mod.

(3) These give <u>all</u> irred. G-mod.

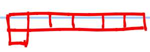
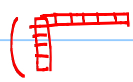
Eg. $\lambda = (d)$ 

$a_{(d)} = \sum_{g \in S_d} g = c_{(d)}$ & $b_{(d)} = 1 \Rightarrow V_{(d)} = \mathbb{C} \cdot c_{(d)} = U$

Eg. $\lambda = (1,1,\dots,1)$ 

$a_{(1,1,\dots,1)} = 1$ $\quad b_{(1,1,\dots)} = \sum_{g \in S_d} \operatorname{sgn}(g) g \Rightarrow V_{(1,\dots,1)} = U'$

In general,

$V_{(}$$_{)} = V_{std.}$ & $V_{(}$$_{)} = \Lambda^s V_{std.}$

Frobenius Formula

$\chi_\lambda : S_\lambda \to GL(V_\lambda) \xrightarrow{Tr} \mathbb{C}$

$\cup$ conj. class

$C_{\underline{i}}$ $\quad \underline{i} = (i_1, i_2, \ldots, i_d)$ w/ $\sum \alpha\, i_\alpha = d$

i_1 1-cycles, i_2 2-cycles etc.

(choose any large k)

$\chi_\lambda(C_{\underline{i}})$

$= \text{coeff. of } x_1^{\lambda_1+k-1} x_2^{\lambda_2+k-2} \cdots x_k^{\lambda_k} \text{ in } \underbrace{\prod_{i<j}(x_i - x_j)}_{\substack{\Delta(x) \\ \text{Vandermonde} \\ \text{det.}}} \cdot \underbrace{\prod_j (x_1^j + x_2^j + \ldots + x_k^j)^{i_j}}_{P_j(x)}$

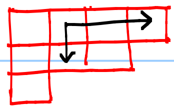
$l_1 = \lambda_1 + k - 1, \; l_2 = \lambda_2 + k - 2, \ldots, l_k = \lambda_k$

$$= \left[\Delta(x) \prod_j P_j(x)^{i_j} \right]_{(l_1, l_2, \ldots, l_k)}$$

Special case (Dimension formula) $\dim V_\lambda = \chi_\lambda(C_{(d)}) = \dfrac{d!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j)$

(Hook length formula) $= d! \,/\, \pi(\text{Hook lengths})$

Eg. $\lambda = (4, 3, 1)$



Hook length:

6	4	3	1
4	2	1	
1			

$\Rightarrow \dim V_\lambda = 8! / 6 \cdot 4 \cdot 3 \cdot 4 \cdot 2$

Claim. V_λ irred.

Notation: $A = \mathbb{C}[S_d]$, $a = a_\lambda$, $b = b_\lambda$, $c = c_\lambda$

Lemma. $c \cdot x\, c \in \mathbb{C} c$

[lemma $\Rightarrow$ claim] $\quad V_\lambda = Ac$

$\Rightarrow \quad c V_\lambda = cAc \subseteq \mathbb{C} c$

$\forall$ submod $W \leq V_\lambda$

$cW \subseteq cV_\lambda \subseteq \mathbb{C} c$

① $cW = \mathbb{C} c \Rightarrow V = Ac \overset{①}{=} AcW \subseteq W \Rightarrow V = W$

② $cW = 0 \Rightarrow W \cdot W \subset \underbrace{V}_{Ac} \cdot W = A \cdot (cW) \overset{②}{=} 0$

$\Rightarrow W = 0$

Refine arguments give $\lambda \neq \mu \Longrightarrow V_\lambda \not\cong V_\mu$

[Pf. of lemma]

- $p \in P \Rightarrow p \cdot a = a \quad (\because a = \sum_{g \in P} g)$
- $p \in P,\ q \in Q \Rightarrow p\, c\, (\mathrm{sgn}\, q)\, q = c$
- $\forall p \in P,\ q \in Q,\quad p\, \gamma\, (\mathrm{sgn}\, q)\, q = \gamma$

  $\Rightarrow \gamma = c$ (up to scaling).

Reason: Write $\gamma = \sum_{g \in S_d} n_g\, g$

$$p\, \gamma\, (\mathrm{sgn}\, q)\, q = \gamma \overset{\star}{\Longleftrightarrow} n_{pgq} = \mathrm{sgn}(q)\, n_g$$

$$\Rightarrow n_{pq} = \mathrm{sgn}(q)\, n_1$$

(i.e. n_g is determined for $g \in PQ$)

If $g \notin PQ$

$\Rightarrow$ (combinatorics) $\exists$ distinct integers, (say 3 & 5) in same row of T (tableau of λ) & in same column of gT (tableau by replace i of T by $g(i)$)

i.e. $p := t = (3\ 5) \in P$

$q := g^{-1} t g \in Q$

$$\Rightarrow g = p g q$$

$$\Rightarrow n_g = n_{pgq} \overset{\text{by } \star}{=} \mathrm{sgn}(\overbrace{q}^{g^{-1}tg})\, n_g = -n_g$$

$$\Rightarrow n_g = 0$$

So $\gamma = \sum_{g \in S_d} n_g\, g$ is completely determined by n_1

$\Rightarrow \gamma = c$ (up to scaling).

- $\forall x,\quad c \cdot x \cdot c = c$ (up to scaling)

Reason: $p\,(c x c)\,(\mathrm{sgn}\, q)\, q$

$= (p a) b\, x\, a (b (\mathrm{sgn}(q)) q) = a b x a b$

$= c x c \quad \forall p \in P\ \&\ q \in Q$

$\Rightarrow c x c = c$ (up to scaling).

Hence the lemma.

Remark: Take $x=1 \Rightarrow c\cdot c = nc \quad \exists n \in \mathbb{C}$.

Consider $F=(-)\cdot c : \underbrace{A}_{\mathrm{Ker}(c)\oplus \underbrace{\mathrm{Im}(c)}_{V_\lambda}} \longrightarrow A$

$$\underbrace{\mathrm{Tr}\,F}_{d! = |S_d|} = \underbrace{\mathrm{Tr}_{\mathrm{Ker}c}\,F}_{0} + \underbrace{\mathrm{Tr}_{\mathrm{Im}c}\,F}_{n\cdot \dim V_\lambda} \qquad (\because c\cdot c = nc)$$

$\left(\because\ c_\lambda = 1 + \cdots\cdots \Rightarrow \text{coeff. of } e_g \text{ in } e_g\cdot c_\lambda \text{ is } 1\right)$

$$\Rightarrow \quad c_\lambda \cdot c_\lambda = \frac{d!}{\dim V_\lambda} c_\lambda$$

§ Proof of Frobenius formula:

Given λ, a partition of d

$$\rightsquigarrow U_\lambda \triangleq A a_\lambda \twoheadrightarrow V_\lambda = A a_\lambda b_\lambda$$

(not irred).

- $\mathbb{C}_{\text{trivial}} \curvearrowleft S_\lambda \triangleq \prod_i S_{\lambda_i} \leq S_d$

s.t. $U_\lambda = \mathrm{Ind}_{S_\lambda}^{S_d} 1$

(Note: $\mathbb{C}_{\text{trivial}} \curvearrowleft H \leq G \Rightarrow \chi_{\mathrm{Ind}1}(\underset{\substack{\cap \\ G}}{C}) = \frac{|G|}{|H|}\cdot\frac{|C\cap H|}{|C|}$ conj. class

$\psi_\lambda : S_d \to GL(U_\lambda) \xrightarrow{\mathrm{Tr}} \mathbb{C}$: char. of U_λ

$$\psi_\lambda(C_{\underline{i}}) = \frac{|S_d|}{|S_\lambda|}\cdot\frac{1}{|C_{\underline{i}}|}|C_{\underline{i}} \cap S_\lambda|$$

$\underline{i} = (i_1\, i_2 \cdots i_d)$
conj. class w/ i_1 1-cycles, i_2 2-cycles, ……

$$= \frac{d!}{\lambda_1!\cdots\lambda_k!}\cdot\frac{1}{\frac{d!}{1^{i_1} i_1!\, 2^{i_2} i_2! \cdots d^{i_d} i_d!}}\cdot\sum_{r_{pq}}\prod_{p=1}^{k}\frac{\lambda_p!}{1^{r_{p1}} r_{p1}! \cdots d^{r_{pd}} r_{pd}!}$$

s.t. $i_q = r_{1q} + r_{2q} + \dots + r_{kq}$

$\lambda_p = r_{p1} + 2r_{p2} + \dots + d\, r_{pd}$

$$= \frac{\cancel{d!}}{\cancel{\lambda_1!\cdots\lambda_k!}}\cdot\frac{1}{\frac{\cancel{d!}}{\cancel{1^{i_1}} i_1!\, \cancel{2^{i_2}} i_2! \cdots \cancel{d^{i_d}} i_d!}}\cdot\sum_{r_{pq}}\prod_{p=1}^{k}\frac{\cancel{\lambda_p!}}{\cancel{1^{r_{p1}}} r_{p1}! \cdots \cancel{d^{r_{pd}}} r_{pd}!}$$

$$= \sum_{r_{pq}}\prod_{q=1}^{d}\frac{i_q!}{r_{1q}!\, r_{2q}! \cdots r_{kq}!} = \left[\prod_{q=1}^{d} P_{i_q}(x)\right]_\lambda =: \left[P^{(\underline{i})}\right]_\lambda$$

For $U_\lambda = A\cdot a_\lambda$, $\psi_\lambda(C_i) = [P^{(i)}]_\lambda$

Aim: $V_\lambda = A\cdot a_\lambda b_\lambda$ $\chi_\lambda(C_i) \overset{?}{=} \underbrace{[\Delta \cdot P^{(i)}]_\ell}_{\text{define } \omega_\lambda(i)}$ (w/ $\ell_j = \lambda_j + k - j$)

Remark: In general,

$$[P]_\lambda = \sum_\mu K_{\mu\lambda} [\Delta P]_{\ell(\mu)}$$ μ, λ : partition of d.

where $K_{\mu\lambda}$ (Kostka #) = # ways fill $T(\mu)$ w/ λ_1 1's λ_2 2's ... λ_k k's s.t. $\longrightarrow$ increasing & $\downarrow$ strictly increasing

- $K_{\lambda\lambda} = 1$; $K_{\mu\lambda} = 0$ if $\lambda > \mu$ (i.e. 1st nonvanishing $\lambda_i - \mu_i$ is positive)

Continue: $\psi_\lambda(C_i) = [P^{(i)}]_\lambda = \sum_\mu K_{\mu\lambda} [\Delta \cdot P^{(i)}]_{\ell(\mu)}$

$$= \omega_\lambda(i) + \sum_{\mu > \lambda} K_{\mu\lambda}\, \omega_\mu(i)$$

$U_\lambda \supseteq V_\lambda$ $\longrightarrow$ $\omega_\lambda \in \mathbb{Z}\langle \chi_\mu\text{'s}\rangle$

($\Rightarrow \psi_\lambda = \sum_\mu n_{\lambda\mu} \chi_\mu$, $n_{\lambda\mu} \geq 0$ & $n_{\lambda\lambda} \geq 1$) Both ω_λ's (combinatrics) & χ_μ's are o.n.

$\Longrightarrow$ $\omega_\lambda = \pm \chi_\mu = \chi_\mu$ (check by hook length formula)

$\Longrightarrow$ $\omega_\lambda = \chi_\lambda$ by induction on $\mu > \lambda$. #.

Remark: Littlewood-Richardson coeff. $N_{\lambda\mu\nu}$

$$V_\lambda \boxtimes V_\mu = \sum_\nu N_{\lambda\mu\nu} V_\nu$$

rep. of S_d S_m S_{d+m}

Chapter 6. Weyl construction.

$V \simeq \mathbb{C}^N$

$$GL(V) \curvearrowright \bigotimes^d V \curvearrowleft S_d \qquad \forall \lambda: \text{partition of } d$$
$$\rightsquigarrow c_\lambda \in \mathbb{C}[S_d]$$

$$\bigotimes^d V \supseteq \underbrace{\mathrm{Im}(c_\lambda)}_{\mathbb{S}_\lambda V}$$

Call: Schur functor, or Weyl constr.

Eg. $\lambda = (d)$ $\boxed{\ \ \ \ \ \ }$ $\Longrightarrow$ $\mathbb{S}_{(d)} V = \mathrm{Sym}^d V$

$\lambda = (1 1 \cdots 1)$ $\Longrightarrow$ $\mathbb{S}_{(1 1 \cdots 1)} V = \Lambda^d V$

Eg. $d = 3$ $\lambda = (2,1)$

(direct check $\Rightarrow$) $\mathbb{S}_{(2,1)} V = \mathrm{Ker}(\Lambda^2 V \otimes V \longrightarrow \Lambda^3 V)$

$$\bigotimes^3 V = \mathrm{Sym}^3 V \oplus \Lambda^3 V \oplus (\mathbb{S}_{(2,1)} V)^{\oplus 2}$$ as irred. decomp. of $GL(V)$-rep.

<u>Theorem</u> (1) $\mathbb{S}_\lambda V$: irred. $GL(V)$-mod.

$$V^{\otimes d} = \bigoplus_\lambda \underset{GL(V)}{\mathbb{S}_\lambda V} \otimes \underset{S_d}{V_\lambda}$$ decomp. as irred. repr.

(2) $$\chi_{\mathbb{S}_\lambda V}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix} = S_\lambda(x_1, \ldots, x_k) \quad \text{Schur poly.}$$

$$\dim \mathbb{S}_\lambda V = \prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

Remark: As decomp. / $GL(V) \times GL(W)$

$$\mathbb{S}_\nu (V \oplus W) = \bigoplus N_{\lambda\mu\nu} (\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W).$$

- $\chi_{\mathrm{Sym}^d V}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix} = H_d(x_1, \ldots, x_k)$ complete symmetric polyn.

  $\chi_{\Lambda^d V}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix} = E_d(x_1, \ldots, x_k)$ elementary symm. polyn.

Recall (alg) $U \curvearrowleft A = \mathbb{C}[G]$

$\Rightarrow$ $B = \mathrm{Hom}_G(U,U)$ commutator alg.

$\underset{\text{Schur}}{=} \bigoplus_i \mathfrak{gl}(n_i)$ (write $U = \bigoplus_i U_i^{\oplus n_i}$ irred. dec. as A-mod.)

$B \curvearrowright U \curvearrowleft A \curvearrowright W \Rightarrow B \curvearrowright U \otimes_A W$

Lemma: (i) $U \otimes_A Ac \longrightarrow Uc$ $\cong$ as B-mod.

(ii) Ac irred A-mod. $\Rightarrow$ Uc irred. B-mod.

(iii) Ac_i's distinct irred. A-mod

$\Rightarrow U \simeq \bigoplus_i (U \otimes_A Ac_i)^{\oplus m_i}$ $m_i = \dim Ac_i$

$\simeq \bigoplus_i (Uc_i)^{\oplus m_i}$ irred. dec. as B-mod.

In our case, $U = V^{\otimes d} \curvearrowleft \mathbb{C}[S_d]$

$B = \mathrm{Hom}_{S_d}(V^{\otimes d}, V^{\otimes d}) \subseteq \mathrm{End}(V^{\otimes d})$

- B is alg. spanned by $\mathrm{End}(V)$.

($\because$ $\mathrm{Sym}^d W$ is spanned by W as alg.).

$\mathbb{S}_\lambda V = Uc_\lambda \overset{\text{lemma}}{\rightsquigarrow}$ Part (1)

$\simeq V^{\otimes d} \otimes_A \underbrace{(Ac_\lambda)}_{V_\lambda}$ isom. as $GL(V)$-mod.

$\underbrace{V^{\otimes d} \otimes_A \underbrace{(A a_\lambda)}_{\underbrace{U_\lambda}_{\bigoplus_\mu K_{\mu\lambda} V_\mu}}}_{\bigoplus_\mu K_{\mu\lambda} \mathbb{S}_\mu V} \simeq \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V$

$\Rightarrow \sum_\mu K_{\mu\lambda} \mathrm{Tr}\, \mathbb{S}_\mu \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix} = H_\lambda(x_1, \dots, x_k)$ ($\checkmark \because \mathrm{Sym}^n V$)

(invertible relat$^{\underline{n}}$) $\Rightarrow \mathrm{Tr}\, \mathbb{S}_\mu(\ddots) = S_\lambda(x_1, \dots, x_k)$ Schur polyn.

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